θ-free graphs: characterization and consequences

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Abstract

For most problems related to perfect matchings, one may restrict attention to matching covered graphs — connected nonempty graphs in which each edge belongs to some perfect matching; Lucchesi and Murty's monograph [Perfect Matchings: A Theory of Matching Covered Graphs, 2024] discusses the extensive literature on them. A cornerstone of this theory is Lovász and Plummer's [Matching Theory, 1986] ear decomposition theorem, which yields interesting open problems in addition to being a fundamental problem-solving tool. We discuss two such open problems below.

By a *bisubdivision* of a graph J, we mean a subdivision wherein each edge is replaced by a path of odd length; any bisubdivision of a matching covered graph (except K_2) is also matching covered. A subgraph H of a graph G is *conformal* if $G - V(H)$ has a perfect matching. The aforementioned ear decomposition theorem of Lovász and Plummer implies that every matching covered graph (except K_2 and even cycles) has a conformal bisubdivision of either θ or of K_4 (possibly both). This immediately leads to two problems: characterize 'θ-free' (matching covered) graphs, and likewise, 'K_4-free' ones. A characterization of planar K_4-free matching covered graphs was obtained by Kothari and Murty [*J. Graph Theory*, 2016]; the nonplanar case is open.

We provide a characterization of θ-free graphs that immediately implies a poly-time algorithm for the corresponding decision problem. Our result and its proof relies heavily on a seminal result due to Edmonds, Lovász and Pulleyblank [*Combinatorica*, 1982] pertaining to the tight cut decomposition theory of matching covered graphs. As corollaries, we prove an upper bound of $2n - 2$ on the size of θ-free graphs, and we also obtain forbidden minor type characterizations of Pfaffian (θ-free) graphs and 3-edge colorable cubic (θ-free) graphs. Finally, we prove that a cubic graph is θ-free if and only if the length of each conformal cycle is a multiple of four. The Petersen graph and K_4 play key roles in our results.